# Electromagnetic Resonances and Q-Factors of Lossy Dielectric Spheres

PETER AFFOLTER AND BALDUR ELIASSON

*Abstract*—**A theoretical and experimental study of the electromagnetic resonances of spheres is presented. In particular, the scattering characteristics of spheres inside rectangular waveguides are investigated at and around the resonant frequencies.**

**The approach is based on the scattering theory developed by Mie in 1908. Mie's theory is valid for scattering of a plane electromagnetic wave by a homogeneous and isotropic sphere of arbitrary diameter. It encompasses both lossless and lossy spheres. Three continuous functions of frequency are presented. They contain information on the resonant frequencies, the *Q*-factors, and the output power losses of the sphere. The effect of losses on the resonant behavior was also studied.**

**The theoretical results were compared to experimental data. The agreement between theory and experiment is excellent. An experimental study of the effect of inhomogenities and irregularities of the sphere's material and shape was also made.**

## Introduction

During our investigations of dielectric resonators of different configurations we have studied in particular the problem of the resonant dielectric sphere. The electromagnetic resonance of a solid sphere has been studied by many investigators [1]–[7]. There is an increasing interest in miniaturizing microwave filters. Some of these filters use resonant spheres. Many high quality single crystals and ceramics of high dielectric constant are now available or are being developed. Considerable effort has been spent on using these for dielectric resonator studies and experiments [8]–[14].

In 1909 Debye deducted the eigenvalue equations for the natural resonant frequencies of free dielectric and metallic spheres [2]. The equations can also be deduced from the theoretical studies of Mie on the scattering of plane waves by spheres [1]. In 1924 Schaefer and Wilmsen did some experiments with short radio waves and compared them to the results of Debye and Mie [3].

In 1938 Richtmyer suggested that a sphere might be used as a dielectric resonator at microwave frequencies [4]. In his paper he calculates the *Q*-factors of high-order modes for relatively small dielectric constant.

In 1967 Gastine *et al.* evaluated Debye's eigenvalue equation numerically [5], [20]. The calculations are valid for a relative dielectric constant in the range 1–100. The losses of the material were not considered. Some of the theoretical results are compared to experimental data. The experimental setup included a sphere symmetrically located inside a rectangular waveguide.

The investigations presented in this paper differ from all the previously mentioned studies in the following aspects. The discussion of the resonant frequencies of the spheres will not be based upon Debye's eigenvalue equation but rather on Mie's scattering theory [15, p. 154]. In this paper we consider spheres made of an arbitrary lossy material. We will also study the effect of the spheres on the fields at frequencies other than the resonant frequencies. Finally we will present a short discussion of the effect a waveguiding structure will have on the resonant frequencies of the sphere. An approximate equation for the power loss ratio of a microwave filter will be given and compared to experimental data.

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The authors are with Brown Boveri & Cie., AG, CH-5401 Baden, Switzerland.

## General Remarks

In all the previously mentioned papers, the discussion is centered around the resonant frequencies of lossless dielectric spheres as derived from Debye's eigenvalue equations. For a certain natural mode number $n$, these have been derived from the condition that the amplitudes $a_n$ and $b_n$ of the modes are equal to infinity. $a_n$ and $b_n$ are the amplitudes of the transverse magnetic (TM) and transverse electric (TE) modes, respectively. The condition of resonance leads to the familiar implicit equations for the resonant frequencies, viz. (see, e.g., [5, p. 695])

$$\text{TM:}\quad \frac{J_{n-1/2}(m\alpha)}{J_{n+1/2}(m\alpha)} = m\cdot\frac{H_{n-1/2}^{(2)}(\alpha)}{H_{n+1/2}^{(2)}(\alpha)} - \frac{n}{\alpha}\cdot\frac{(m^2-1)}{m} \tag{1}$$

$$\text{TE:}\quad \frac{J_{n-1/2}(m\alpha)}{J_{n+1/2}(m\alpha)} = \frac{1}{m}\cdot\frac{H_{n-1/2}^{(2)}(\alpha)}{H_{n+1/2}^{(2)}(\alpha)} \tag{2}$$

where $\alpha = 2\pi r/\lambda_2 = k_2 r$, $r$ is the radius of the sphere, $\lambda_2$ is the wavelength in the medium surrounding the sphere, $k_2$ the wavenumber, and $m$ the sphere's relative index of refraction. $J_\nu$, $N_\nu$, and $H_\nu^{(2)} = J_\nu - iN_\nu$ are the Bessel, Neumann, and Hankel functions, respectively. The solutions of (1) and (2) imply that a scattered field exists without any incident field. The real physical interpretation of these resonances is not immediately evident as the solutions of (1) and (2) lead to complex frequencies. The solution becomes especially difficult to obtain and interpret when the sphere has a complex index of refraction.

We will study the general resonant properties of a sphere which scatters an incident plane wave of infinite extent. As a particular application, we will consider the case of a sphere symmetrically located in a hollow waveguide.[1] The combination of waveguide and sphere acts as a narrow band-stop filter. At the resonant frequencies, practically no power is transmitted. We will establish some approximate expressions for the performance of such filters.

Our approach to the problem is based on the well-known theory of Mie [16, p. 39]. The Mie theory describes the scattering of a plane electromagnetic wave by a sphere made of

[1] The modes of propagation in and on many microwave waveguide structures are combinations of plane waves propagating in various directions (see e.g., [17, p. 106]). The waves inside the waveguide are of finite extent. To correctly solve the scattering problem inside the waveguide, one would have to expand the field in the form of the general solution of the scalar wave equation in spherical coordinates. One would also have to fulfill the boundary conditions along the surface of the sphere. This approach falls outside the scope of this paper.

an arbitrary, isotropic, and homogeneous material. For arbitrary free spheres, the resonant frequencies are obtained as those values of the frequency $\omega$ for which the real part of the amplitudes $a_n$ and $b_n$ are at a maximum. For lossless spheres of a very large dielectric constant we get resonant frequencies which are almost the same as those defined in (1) and (2). In the lossless case the resonances in the Mie theory occur for all values of $a_n = b_n = (1.0 + i\,0.0)$. This condition leads to the following equations for the resonant frequencies:

$$\text{TM:}\quad \frac{J_{n-1/2}(m\alpha)}{J_{n+1/2}(m\alpha)} = m\cdot\frac{N_{n-1/2}(\alpha)}{N_{n+1/2}(\alpha)} - \frac{n}{\alpha}\cdot\frac{(m^2-1)}{m} \quad (3)$$

$$\text{TE:}\quad \frac{J_{n-1/2}(m\alpha)}{J_{n+1/2}(m\alpha)} = \frac{1}{m}\cdot\frac{N_{n-1/2}(\alpha)}{N_{n+1/2}(\alpha)}. \quad (4)$$

Equations (3) and (4) are very similar to (1) and (2). In this paper we are mainly interested in the solution of the above equation for $m \gg 1$. The corresponding solution for $\alpha = \alpha_{\text{res}}$ will then be small, i.e., $\alpha_{\text{res}} < 1$. For such small arguments the Neumann functions are orders of magnitude larger than the corresponding Bessel functions on the right-hand side of (1) and (2). The solutions of these equations will, therefore, not be very different from the corresponding solutions to (3) and (4). For smaller $m$ the situation is different. The solutions of (1) and (2) may differ considerably from those of (3) and (4). The Mie resonant frequencies obtained from (3) and (4) seem to conform more closely to the real physical nature of the problem as the incident wave has been taken into account.

As a first-order approximation we will consider the plane waves inside the waveguide to be of infinite extent. Thus, at frequencies close to the resonant frequency, the expression for the power scattered by the sphere will be only approximately valid. The resonant frequencies of the sphere inside the waveguide will differ only slightly from the ones observed in free space. The width of the resonant peak allows one to estimate the loaded $Q$-factor of the system.

By dividing the scattered power by the power incident on the sphere, one gets a dimensionless function of frequency, $R(\omega)$. This function contains all the information regarding the resonant frequencies.

Another continuous function, $Q(\omega)$, will be defined. It is equal to $\omega$ times the energy stored within the sphere divided by the power scattered and absorbed by the sphere. At all resonant frequencies its value is equal to the unloaded $Q$-factor of the free sphere.

For the filter itself, we will deduce an approximate expression for the power loss ratio, i.e., the ratio of the powers available at the filter output and input, respectively. This quantity $P(\omega)$ which is a function of $R(\omega)$ can also be measured and thus compared to the theoretical values.

All the calculations needed are easily performed on a computer once a program supplying the Mie parameters of the scattering process is available.

The approach presented in this paper has various advantages over previously made calculations. The difficulties associated with the interpretation of the problem in terms of single natural modes do not arise here. The Mie theory allows expressions for the total scattered field to be derived. These expressions include combinations of all excited modes. The question of resonance and $Q$-factors can be answered quite simply by inspection of the power scattered and absorbed by the sphere as a function of frequency. The effect of absorption on the resonant frequencies can be studied easily.

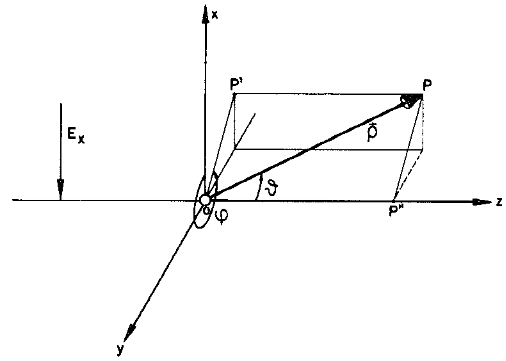


Fig. 1. Geometry for Mie scattering. Incident wave travels along positive $z$ axis with electric vector polarized along $x$ axis. A sphere with radius $r$ is centered at the origin. Direction of scattered wave is defined by polar angles $\theta$ and $\phi$.

In the following section we will give a more detailed account of the application of the Mie theory to the problem.

## Application of the Theory of Mie

The Mie theory describes the scattering of a plane electromagnetic wave by an isotropic and homogeneous sphere. The scattered field vectors can be decomposed into three components, $E_\theta$, $E_\phi$, and $E_\rho$ (see Fig. 1).

The incident field is assumed to be polarized along the $x$ axis. The sphere is located at the origin of the coordinate system. The $E_\theta$-component lies in the $POP''$ plane (the scattering plane) and the $E_\phi$-component is perpendicular to that plane. The component $E_\rho$ is pointed along the radius vector $\boldsymbol{\rho}$. There are two types of modes excited by the incident wave, viz. the TM modes and the TE modes. The TM modes have no magnetic vector along $\boldsymbol{\rho}$, and analogously the TE modes have no electric vector along $\boldsymbol{\rho}$ (see e.g., [5, p. 698]). The radial components decrease as $1/\rho^2$ whereas the transverse components decrease as $1/\rho$. For distances larger than two or three wavelengths, the radial components can be ignored, and both modes then lie in a plane perpendicular to the radius vector.

In the far-field zone, the total electric field is given by

$$E_\phi = -i\,\frac{\exp(-ik_2\rho)}{k_2\rho}\cdot S_\phi\cdot \sin\phi \quad (5)$$

$$E_\theta = i\,\frac{\exp(-ik_2\rho)}{k_2\rho}\cdot S_\theta\cdot \cos\phi \quad (6)$$

where

$$S_\phi = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\left[a_n\pi_n(\cos\theta) + b_n\tau_n(\cos\theta)\right] \quad (7)$$

$$S_\theta = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)}\left[a_n\tau_n(\cos\theta) + b_n\pi_n(\cos\theta)\right]. \quad (8)$$

The coefficients $a_n$ and $b_n$ are the amplitudes of the $\text{TM}_n$ and $\text{TE}_n$ modes, respectively. They are functions of the relative index of refraction $m$ of the sphere and the size parameter $\alpha$ according to the following formulas:

$$a_n = \frac{\psi_n(\alpha)\cdot\psi_n'(m\alpha) - m\cdot\psi_n(m\alpha)\psi_n'(\alpha)}{\zeta_n(\alpha)\cdot\psi_n'(m\alpha) - m\cdot\psi_n(m\alpha)\zeta'(\alpha)}$$

$$b_n = \frac{m\psi_n(\alpha)\psi_n'(m\alpha) - \psi_n(m\alpha)\psi_n'(\alpha)}{m\zeta(\alpha)\psi_n'(m\alpha) - \psi_n(m\alpha)\zeta'(\alpha)}.$$

The function $\zeta_n(x)$ is equal to $\psi_n(x)+i\chi_n(x)$, where $\psi_n(x)$ and $\chi_n(x)$ are the Ricatti–Bessel functions defined in [16, p. 43]. A corresponding solution with coefficients $c_n$ and $d_n$ can be found inside the sphere. Here the radial fields do not vanish. This solution is needed to calculate the energy stored inside the sphere. Formulas for $c_n$ and $d_n$ are given in [16, p. 45].

The functions $\pi_n(\cos\theta)$ and $\tau_n(\cos\theta)$ depend only on the scattering angle $\theta$. They are defined in terms of the associated Legendre polynomials $P_n^{(1)}(\cos\theta)$ (see [16, p. 47]).

The coefficients $a_n$, $b_n$, $c_n$, and $d_n$ are oscillating functions of the index $n$. For $n$ somewhat larger than $\alpha$ they decrease rapidly as a function of $n$. The sums in (7) and (8) must therefore only be summed up to some $N$. This $N$ specifies the number of all spherical modes which have been excited by the incident wave.

Three important parameters which also specify the scattering process are the so-called cross sections of extinction, scattering, and absorption. The cross section for extinction multiplied by the incident intensity (i.e., power per unit area) is equivalent to the power abstracted from the incident beam. Likewise do the cross sections of scattering and absorption specify the power lost due to scattering and absorption, respectively. The extinction cross section is equal to the sum of the scattering and absorption cross sections.

The cross section of extinction is given by

$$C(\omega)=\sum_{n=1}^{\infty}(C_n{}^{\mathrm{TM}}+C_n{}^{\mathrm{TE}})=\frac{\lambda_2^2}{2\pi}\sum_{n=1}^{\infty}(2n+1)(a_n'+b_n') \quad (9)$$

where $a_n'$ and $b_n'$ are the real parts of $a_n$ and $b_n$, respectively. The energy stored, $W$, consists of two parts, $W_1$ and $W_2$. $W_1$ is the energy stored inside the sphere, and $W_2$ is the reactive energy stored outside the sphere. Thus we have

$$W=W_1+W_2$$

where

$$W_1=\frac{1}{4}\int_0^{2\pi}\int_0^{\pi}\int_0^{r}(\epsilon_1'|E|^2+\mu_1|H|^2)\rho^2\sin\theta\,d\phi d\theta d\rho \quad (10)$$

and

$$\begin{aligned}W_2&=\sum_{n=1}^{\infty}(W_{2n}{}^{\mathrm{TM}}+W_{2n}{}^{\mathrm{TE}})\\&=\frac{1}{4}\int_0^{2\pi}\int_0^{\pi}\int_r^{\infty}(\epsilon_2[|E|^2-|E_F|^2]\\&\quad+\mu_2[|H|^2-|H_F|^2])\rho^2\sin\theta\,d\phi d\theta d\rho\\&=\frac{\pi^2}{4}\cdot\frac{\epsilon_2}{k_2^3}\sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)(2n+1)\\&\quad\cdot\Big[(n-\alpha^2)(J_{n+1/2}^2(\alpha)+N_{n+1/2}^2(\alpha))\\&\quad-\alpha^2(J_{n-1/2}^2(\alpha)+N_{n-1/2}^2(\alpha))\\&\quad+2n\alpha(J_{n+1/2}(\alpha)J_{n-1/2}(\alpha)\\&\quad+N_{n+1/2}(\alpha)N_{n-1/2}(\alpha))+\frac{4}{\pi}\alpha\Big].\end{aligned}$$

$\epsilon_1'$ is the real part of the dielectric constant of the sphere. The permeability $\mu_1=\mu_2$ is taken as real. $E_F$ and $H_F$ are the far-field components of the scattered field given by (5) and (6). For all cases which are considered in this paper (i.e., large dielectric constants), the reactive energy $W_2$ is always much smaller than the energy $W_1$.[2] For the TM modes we have $(W_2/W_1)_{\mathrm{TM}}\to 0$ for $m\to\infty$. For the TE modes the situation is different. Here the ratio $(W_2/W_1)$ converges towards a limit, viz., $(W_2/W_1)_{\mathrm{TE}_{nr}}\to n/([(m\alpha)_{nr}]^2-n)$ for $m\to\infty$. $(m\alpha)_{nr}$ is the $r$th zero of the function $J_{n-1/2}(m\alpha)$.

The result of the integration in (10) is

$$W_1(\omega)=\sum_{n=1}^{\infty}(W_{1n}{}^{\mathrm{TM}}+W_{1n}{}^{\mathrm{TE}})$$

where

$$\begin{aligned}W_{1n}{}^{\mathrm{TM}}&=\frac{1}{4}\cdot\frac{\pi^2}{|k_1|^3}\cdot|c_n|^2\cdot(2n+1)\cdot\epsilon_2\\&\quad\cdot\Big\{-n\cdot(m'^2-m''^2)\cdot|J_{n+1/2}(m\alpha)|^2\\&\quad-\alpha\cdot\frac{m''^2}{m'}(m'^2+m''^2)\Big[\mathrm{Re}\{J_{n+1/2}(m\alpha)\}\\&\quad\cdot\mathrm{Re}\{J_{n-1/2}(m\alpha)\}+\mathrm{Im}\{J_{n+1/2}(m\alpha)\}\\&\quad\cdot\mathrm{Im}\{J_{n-1/2}(m\alpha)\}-\left(\frac{m'}{m''}\right)^3\\&\quad\cdot\Big[\mathrm{Im}\{J_{n+1/2}(m\alpha)\}\cdot\mathrm{Re}\{J_{n-1/2}(m\alpha)\}\\&\quad-\mathrm{Re}\{J_{n+1/2}(m\alpha)\}\,\mathrm{Im}\{J_{n-1/2}(m\alpha)\}\Big]\Big]\Big\} \quad (11)\end{aligned}$$

$$W_{1n}{}^{\mathrm{TE}}=\frac{W_{1n}{}^{\mathrm{TM}}(d_n)}{m'^2-m''^2} \quad (12)$$

$$k_1=\frac{2\pi\cdot(m'+im'')}{\lambda_2}=k_2(m'+im'').$$

$k_2$ and $\epsilon_2$ are the wavenumber and dielectric constant, respectively, of the lossless medium surrounding the sphere. $m'$ and $m''$ are the real and imaginary parts of the relative complex index of refraction of the sphere, i.e., $\epsilon_1=\epsilon_1'+i\epsilon_1''=\epsilon_2\epsilon_r=\epsilon_2 m^2=\epsilon_2(m'+im'')^2$. If $m''$ is equal to zero, (11) and (12) can be written in the following simpler form:

$$\begin{aligned}W_{1n}{}^{\mathrm{TM}}&=\frac{\pi^2\cdot|c_n|^2\cdot(2n+1)}{4k_1^3}\cdot\epsilon_2\cdot m'^2\\&\quad\cdot[((m'\alpha)^2-n)\cdot J_{n+1/2}^2(m'\alpha)\\&\quad+(m'\alpha)^2\cdot J_{n-1/2}^2(m'\alpha)\\&\quad-2n\cdot\alpha\cdot m'\cdot J_{n+1/2}(m'\alpha)J_{n-1/2}(m'\alpha)] \quad (13)\end{aligned}$$

$$W_{1n}{}^{\mathrm{TE}}=\frac{W_{1n}{}^{\mathrm{TM}}(d_n)}{m'^2}. \quad (14)$$

We get the function $R(\omega)$ from (9) by dividing $C(\omega)$ by $\pi r^2$, as follows:

$$R(\omega)=\frac{2}{\alpha^2}\sum_{n=1}^{\infty}(2n+1)(a_n'+b_n'). \quad (15)$$

[2] For instance, for $\epsilon_r=100$, the evaluation of the above equations for $W_1$ and $W_2$ gives at resonance for the $\mathrm{TM}_{11}$ and $\mathrm{TE}_{11}$ modes, respectively, $(W_2/W_1)_{\mathrm{TM}_{11}}=0.017$ and $(W_2/W_1)_{\mathrm{TE}_{11}}=0.128$.

$R(\omega)$ is identical to the efficiency factor for extinction defined in [16, p. 50]. It attains a maximum value at each resonant frequency.

The customary definition of the unloaded $Q$-factor for a resonant cavity is $Q_{on}$ = (energy stored times $\omega$)/(power dissipated). This value of $Q_{on}$ is only defined for the resonant frequency $\omega$ and the mode number $n$. Our continuous function $Q(\omega)$ is defined as follows:

$$Q(\omega) = 2\omega\cdot\left(\frac{\mu_2}{\epsilon_2}\right)^{1/2}\frac{W(\omega)}{C(\omega)} = 2\omega\cdot\left(\frac{\mu_2}{\epsilon_2}\right)^{1/2} \cdot \frac{\sum_{n=1}^{\infty}(W_{1n}{}^{TM}+W_{2n}{}^{TM}+W_{1n}{}^{TE}+W_{2n}{}^{TE})}{\sum_{n=1}^{\infty}(C_n{}^{TM}+C_n{}^{TE})}. \tag{16}$$

This function has the property that at each resonant frequency it attains the value of the corresponding $Q_{on}$. $Q(\omega)$ is like $R(\omega)$ solely a function of the material and radius of the sphere and the wavelength of the incident radiation. It is no measure for the quality of a sphere as a microwave resonator in the setup considered in this paper.

We define the power loss ratio $P(\omega)$ as the ratio of the output of the filter to its input when the filter is terminated in a matched load [17, p. 403]. For a sphere placed within a rectangular waveguide of width $a$ and height $b$, the function $P(\omega)$ is approximately equal to ($H_{10}$-mode propagation)

$$P(\omega) = 1 - \pi\cdot\frac{r^2}{a\cdot b}\cdot\cos\chi\cdot R(\omega) \tag{17}$$

where $\cos\chi = (1-(\lambda_2/2a)^2)^{1/2}$.[3] The plane waves inside the waveguide propagate in directions inclined at an angle $\chi$ with respect to the vertical walls of the waveguide [17, p. 106]. As we have mentioned earlier, the expression in (17) is an approximation as it does not take the finite width of the field fully into account. As we will see later, it is a convenient and relatively accurate measure of the width of the resonance peaks of this measurable quantity.

In the following section we will give a brief discussion of the functions $R(\omega)$ and $Q(\omega)$.

## Discussion

In Fig. 2 a graph of the function $R(\omega)$ is given for the arbitrarily chosen parameters $\epsilon_r = 100$ and $\epsilon_r = 100\ (1-i\ 0.01)$, respectively ($\epsilon_2=\epsilon_0$, $\mu_1=\mu_2=\mu_0$). The frequency $f=\omega/2\pi$ varies in the range 7/15 GHz. On the lower horizontal axis, the type of resonance is indicated ($TM_{nr}$ or $TE_{nr}$). $n$ is the mode number and $r$ is the order of the resonance.

$R(\omega)$ has a well defined peak at each resonant frequency $\omega_{res}$. The widths of the peaks at the 3-dB points vary within wide limits. The width generally decreases with increasing mode number. For lossless dielectrics, the maximum value of $R(\omega)$ at each resonant frequency is practically independent of the dielectric constant. It is approximately given by the value of $R(\omega)$ for the resonant mode, i.e.,

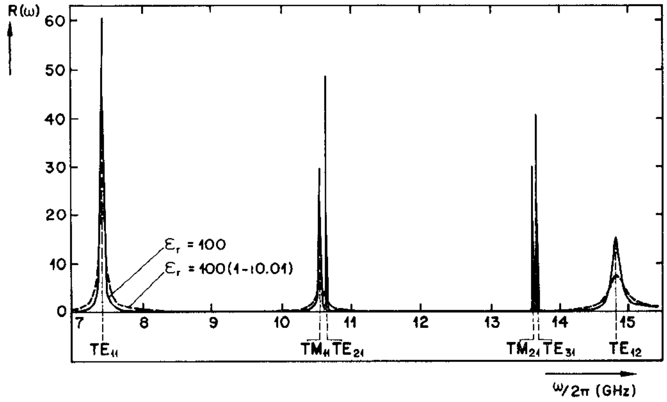


Fig. 2. The efficiency factor for extinction for $\epsilon_r = 100$ and $\epsilon_r = 100\ (1-i\ 0.01)$ versus frequency $f=\omega/2\pi$. Mode identification is indicated below the frequency axis. Sphere diameter = 0.4000 cm.

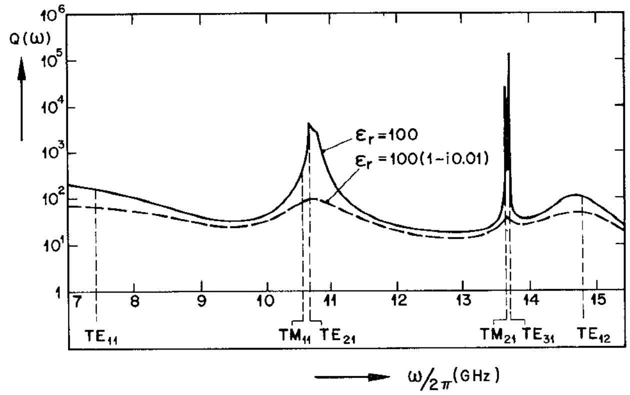


Fig. 3. Continuous $Q(\omega)$ function for $\epsilon_r = 100$ and $\epsilon_r = 100\ (1-i\ 0.01)$ versus frequency $f=\omega/2\pi$. Mode identification is indicated below the frequency axis. Sphere diameter = 0.4000 cm.

$$R(\omega) \cong \frac{4n+2}{\alpha_{res}{}^2} = (4n+2)\cdot\frac{c^2}{\omega_{res}{}^2\cdot r^2} \tag{18}$$

where $c$ is the velocity of light in the medium surrounding the sphere.

The corresponding graph for $Q(\omega)$ is shown in Fig. 3. It has been calculated for a radius of $2\times10^{-3}$ m.

The values of the coefficients $c_n$ and $d_n$ for lossless dielectrics [(13) and (14)] at the resonant frequencies are equal to $i(m')^{-1/2}N_{n+1/2}(\alpha)/J_{n+1/2}(m'\alpha)$ and $i(m')^{3/2}N_{n+1/2}(\alpha)/J_{n+1/2}(m'\alpha)$, respectively. By inserting the above values into (13) and (14) and subsequently inserting these expressions along with (3), (4), and (18) into (16), we get for $Q(\omega_{res})$:

$$\begin{aligned} Q_{res}{}^{TM} = \frac{\pi}{4}\Bigg[&\left(n^2\left(m^2-\frac{1}{m^2}\right)-\frac{n}{m^2}+n\right)N_{n+1/2}{}^2(\alpha) \\ &+\alpha^2(m^2-1)N_{n-1/2}{}^2(\alpha) \\ &-2n\alpha(m^2-1)N_{n+1/2}(\alpha)N_{n-1/2}(\alpha) \\ &+\frac{4}{\pi}\alpha+(n-\alpha^2)J_{n+1/2}{}^2(\alpha) \\ &-\alpha^2J_{n-1/2}{}^2(\alpha)+2n\alpha J_{n+1/2}(\alpha)J_{n-1/2}(\alpha)\Bigg] \end{aligned} \tag{19}$$

[3] Equation (17) is approximately valid as long as the scattering cross section of the sphere is smaller than the waveguide cross section. This is identical to the condition $P(\omega) > 0$. The range of validity of (17) can also be established in another way. Most of the reactive energy $W_2(r')$ which is stored outside the sphere is concentrated within a concentric sphere of radius $r'$ surrounding the resonator. When this imaginary concentric sphere is wholly contained within the waveguide, then (17) will be a fairly good approximation. For instance, if $\epsilon_r = 100$, then 90 percent of $W_2(\infty)$ at resonance is contained within a sphere of radius $r' = 3.75\ r$.

$$Q_{res}^{TE} = \frac{\pi}{4}\Bigg[\alpha^2(m^2-1)N_{n+1/2}^2(\alpha) + (n-\alpha^2)J_{n+1/2}^2(\alpha) - \alpha^2 J_{n-1/2}^2(\alpha) + 2n\alpha J_{n+1/2}(\alpha)J_{n-1/2}(\alpha) + \frac{4}{\pi}\alpha\Bigg]. \quad (20)$$

The parameter $\alpha$ is equal to $\alpha = \alpha_{res}^{TM} = \omega_{res}^{TM}\ r/c$ and $\alpha = \alpha_{res}^{TE} = \omega_{res}^{TE}\ r/c$, respectively. In the lossy case, the function $Q(\omega)$ can be calculated from (11) and (12).

## Experiments

As an illustration of the theory developed and discussed in the previous chapters, some results of measurements will be given in this section. The resonant behavior of a sphere placed inside a rectangular waveguide will be studied. In order to minimize the splitting up of the modes [5, p. 700], the measurements were made with a sphere of a single crystal, $SrTiO_3$. All measurements were made at $X$-band frequencies. The sphere's diameter was $2.000 \pm 0.001$ mm. Its relative dielectric constant as determined by the frequency of the $TM_{11}$ mode was $\epsilon_r' = 324.4$ at room temperature. Inside the waveguide, the sphere was positioned by means of four thin Teflon tabs. Several measurements revealed that the position of the sphere has practically no influence on the resonant frequency. Furthermore, no splitting up of the modes into two of three separate lines was observed when varying the position of the sphere inside the waveguide. This splitting up does occur when using spheres made of polycrystalline $SrTiO_3$ and $TiO_2$. The lines can be as far apart as 0.1 GHz. The appearance of frequency splitting when using polycrystalline spheres leads us to believe that the macroscopic isotropy and/or homogeneity is slightly altered when the spheres are made by using the pressing and sintering process. Note that $SrTiO_3$ and $EuTiO_3$ are the only titanates which show perfectly cubic structures at room temperature [18, p. 415].

In order to avoid difficulties due to the very high temperature dependency of the dielectric constant, the least possible amount of RF power has to be used when measuring [19, p. 241].

At room temperature and $X$-band frequencies, the loss tangent $\tan\delta = \epsilon_r''/\epsilon_r'$ of single crystal $SrTiO_3$ is $5 \cdots 10 \times 10^{-4}$ [19, p. 241]. Our theory shows that loss tangents of this order considerably decrease the resonance peak of the third excited mode, i.e., the $TE_{21}$ mode. The expected depth of resonance is of the order of the ripple of a directional coupler. In order to obtain minimum ripple of the power incident on the sample sphere, the experimental setup shown in Fig. 4 was used for making all the measurements.

Fig. 5(a) shows a plot of the results obtained by the Mie computer program for $P(\omega)$ (17) at frequencies between 7.9 and 12.4 GHz. The parameters chosen were $\epsilon_r' = 324.4$ and $\tan\delta = 7 \times 10^{-4}$. Three modes occurring within the $X$ band are clearly shown by the calculations. The two relevant frequency domains of Fig. 5(a) are illustrated in Fig. 5(b), where $\tan\delta$ was varied between $1 \times 10^{-2}$ and $1 \times 10^{-4}$. For $\tan\delta = 1 \times 10^{-3}$, the $TE_{21}$ resonance practically disappears, while the $TE_{11}$ resonance can still be identified for $\tan\delta = 1 \times 10^{-1}$. The resonant frequencies of the different modes are practically independent of $\tan\delta$ for $1 \times 10^{-2} > \tan\delta > 1 \times 10^{-4}$.

The results of our measurements are shown in Fig. 6(a). The three resonances predicted by the theory are clearly visible. In analogy to Fig. 5(b), Fig. 6(b) shows the relevant frequency domains with the same bandwidth of 0.2 GHz. As

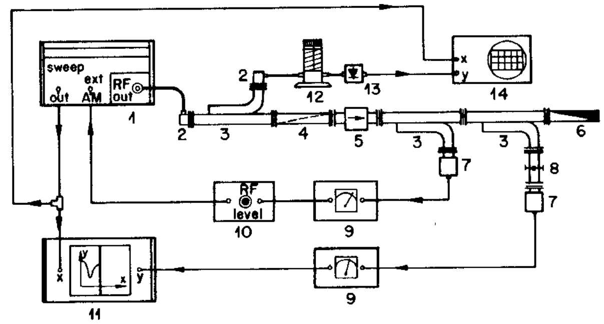


Fig. 4. Experimental setup ($X$ band, WR-90): 1) sweep generator, 2) coax to waveguide adapter, 3) 10-dB directional coupler, 4) twist 90°, 5) ferrite isolator, 6) termination, 7) thermistor mount, 8) sample sphere under test, 9) power meter, 10) leveling amplifier, 11) recorder, 12) frequency meter, 13) crystal detector, and 14) oscilloscope.

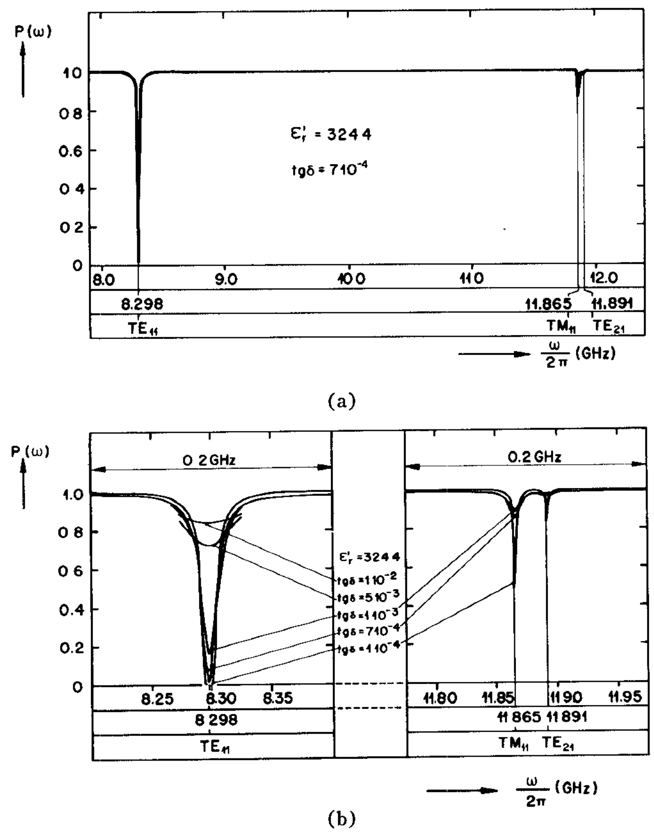


Fig. 5. (a) Theoretical power loss ratio $P(\omega)$ for $\epsilon_r = 324.4\,(1 - i0.0007)$ versus $X$-band frequencies. Sphere diameter = 0.2000 cm. (b) Relevant frequency domains of theoretical power loss ratio $P(\omega)$ for different imaginary parts of $\epsilon_r$. Sphere diameter = 0.2000 cm.

the incident power varies less than 0.5 dB over the whole $X$ band, we were able to use the transmitted power $P_T$ directly for $P_{out}$ without having to form the power loss ratio $P_{out}/P_{in}$.

## Discussion

The measured values agree very well with the calculations. The deviations of the resonant frequencies are of the order of 0.1 percent. Note that a difference in the relative dielectric constant $\epsilon_r'$ of 0.1 percent produces a deviation of 0.05 percent. A relative change of the sphere's diameter of 0.1 percent also produces a deviation of 0.1 percent. It can be concluded that the measurements were done on a practically perfect and isotropic sphere made of a single crystal remarkably free of internal mechanical stress.

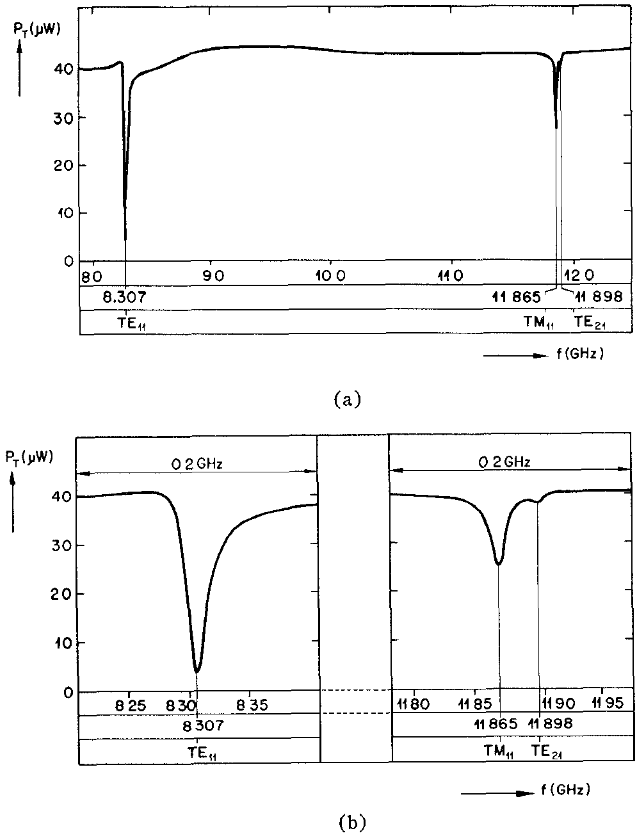


Fig. 6. (a) Recorder plot of transmitted power $P_T$ versus $X$-band frequencies. A $SrTiO_3$ single crystal sphere with a diameter of 0.2000 ± 0.0001 cm was mounted in a WR-90 waveguide. (b) Relevant frequency domains of transmitted power $P_T$. A $SrTiO_3$ single crystal sphere with a diameter of 0.2000 ±0.0001 cm was mounted in a WR-90 waveguide.

The asymmetry of the measured $TE_{11}$ resonance curve is probably caused by the influence of the detector backscatter and the waveguide walls.

The response of the $TM_{11}$ and $TE_{21}$ modes is stronger than predicted by theory. This is probably caused by the field concentrations which are produced by the Teflon carriers in the vicinity of the sphere.

As already suggested by Gastine *et al.* [5, p. 700] and Yu [7, p. 724], experiments as described here are ideally suited for determining high relative dielectric constants $\epsilon_r'$ and associated low-loss tangents tan $\delta$.

The shape and depth of the resonance curves shown in Fig. 5(a) and (b) can, as mentioned in the theoretical section, be used to calculate the loaded $Q$-factors, $Q_{\text{loaded}}$, of the system. Furthermore, by analyzing the splitting up of one or more modes, one can obtain a measure for the dielectrical isotropy of the sample under examination.

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## References


[1] G. Mie, "Beiträge zur Optik trüber Medien," *Ann. Phys.* (Leipzig), vol. 25, pp. 377–445, 1908.
[2] P. Debye, "Der Lichtdruck auf Kugeln von beliebigem Material," *Ann. Phys.* (Leipzig), vol. 30, pp. 57–136, 1909.
[3] C. Schaefer and K. Wilmsen, "Ueber die elektrischen und magnetischen Eigenschwingungen dielektrischer und metallischer Kugeln," *Z. Phys.*, pp. 345–354, 1924.
[4] R. D. Richtmyer, "Dielectric resonators," *J. Appl. Phys.*, vol. 10, pp. 391–398, June 1939.
[5] M. Gastine, L. Courtois, and J. L. Dormann, "Electromagnetic resonances of free dielectric spheres," *IEEE Trans. Microwave Theory Tech.* (*1967 Symposium Issue*), vol. MTT-15, pp. 694–700, Dec. 1967.
[6] I. Wolff, "Die elektromagnetischen Eigenschwingungen einer offenen Ferritkugel," *Arch. Elek. Übertragung*, vol. 23, no. 11, pp. 561–569, 1969.
[7] J. S. Yu, "A resonance method for measurement of large dielectric constant with small loss," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-17, pp. 724–726, Sept. 1969.
[8] A. Okaya, "The rutile microwave resonator," *Proc. IRE* (Corresp.), vol. 48, p. 1921, Nov. 1960.
[9] A. Okaya and L. F. Barash, "The dielectric microwave resonator," *Proc. IRE*, vol. 50, pp. 2081–2092, Oct. 1962.
[10] S. B. Cohn, "Microwave bandpass filters containing high-$Q$ dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 218–227, Apr. 1968.
[11] T. D. Iveland, "Dielectric resonator filters for application in microwave integrated circuits," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Microwave Integrated Circuits*), vol. MTT-19, pp. 643–652, July 1971.
[12] H. M. Müller, "Dielektrische Resonatoren und ihre Anwendungen als Mikrowellenfilter," *Z. Angew. Phys.*, vol. 24, no. 3, pp. 142–147, 1968.
[13] S. Fiedziuszko and A. Jeleński, "The influence of conducting walls on resonant frequencies of the dielectric microwave resonator," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-19, pp. 778–779, Sept. 1971.
——, "Double dielectric resonator," *ibid.*, pp. 779–781.
[14] I. Wolff, "Offene Mikrowellenresonatoren," *Nachritentech. Z.*, vol. 24, no. 6, pp. 299–306, 1971.
[15] H. C. van de Hulst, *Light Scattering by Small Particles*. New York: Wiley, 1957.
[16] M. Kerker, *The Scattering of Light*. New York: Academic, 1969.
[17] R. E. Collin, *Foundations for Microwave Engineering*. New York: McGraw-Hill, 1966.
[18] H. J. Martin, *Die Ferroelektrika*. Leipzig: Akademische Verlagsgesellschaft Geest & Portig K. G., 1964.
[19] R. O. Bell and G. Rupprecht, "Measurements of small dielectric losses in material with a large dielectric constant at microwave frequencies," *IRE Trans. Microwave Theory Tech.*, vol. MTT-9, pp. 239–242, May 1961.
[20] M. Gastine, "Resonances electromagnetiques d'echantillons dielectriques spheriques," thesis, Faculté des Sciences d'Orsay, France, Nov. 1967.